

AMMANN TILINGS IN SYMPLECTIC GEOMETRY

FIAMMETTA BATTAGLIA AND ELISA PRATO

Abstract

The purpose of this article is to view Ammann tilings from the perspective of symplectic geometry.

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Introduction

Our aim is to explore the connection between *Ammann tilings* and symplectic geometry. Ammann tilings are 3D analogues of Penrose rhombus tilings, made of two kinds of tiles: an *oblate* and a *prolate* rhombohedron having same edge lengths. The basic idea underlying this connection is that we are able to associate to each rhombohedron in the tiling a compact connected symplectic *quasifold* in a natural way. Quasifolds generalize manifolds and orbifolds and were first introduced by the second-named author in [6]; locally they are quotients of a manifold modulo the smooth action of a discrete group. The way that we associate a quasifold to a given rhombohedron is by applying a generalization to simple nonrational polytopes [6] of the Delzant construction for simple rational polytopes [3]. We remark that, while each separate rhombohedron in the tiling is a simple rational polytope and could be viewed as a standard symplectic manifold using the Delzant construction, all of the rhombohedra in the tiling are not simultaneously rational with respect to the same lattice. Since we are interested in obtaining a global symplectic view of the tiling we replace the notion of lattice with the more general notion of *quasilattice*: rationality gets then replaced by *quasirationality*. Given an Ammann tiling, we introduce a quasilattice Q having the property that each rhombohedron of the tiling is *quasirational* with respect to Q . We then apply the generalized Delzant procedure simultaneously to each rhombohedron and we show that there is only one symplectic quasifold, M_b , associated to each of the oblate rhombohedra of the tiling and one symplectic quasifold, M_r , associated to each of the prolate rhombohedra (Theorem 3.1). Both quasifolds are globally the quotient of a manifold, actually the product of three 2-spheres, modulo the action of a discrete group. Furthermore, each quasifold is endowed with the effective Hamiltonian action of the *quasitorus* \mathbb{R}^3/Q ; the oblate and prolate rhombohedra are then obtained as images of the respective moment mappings. As it turns out, the two quasifolds M_b and M_r are diffeomorphic but not symplectomorphic (Theorem 4.1).

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The initial motivation of our work on nonperiodic tilings was the realization that applying the generalized Delzant procedure to the tiles provided a very nice way to obtain new low dimensional examples of symplectic quasifolds with symmetries. We refer the reader to [1, 2] for our previous work on Penrose rhombus and kite and dart tilings; the kite, for example, yields an interesting 4-dimensional quasifold that is not a global quotient of a manifold modulo the action of a discrete group.

We remark that Ammann tilings, which were introduced in the 70's [7], later turned out to be crucial in the study of *quasicrystals* with icosahedral symmetry. Quasicrystals are special alloys having discrete nonperiodic diffraction patterns that were discovered by Shechtman et al. [8] in 1982. They have atomic arrangements with symmetries that are not allowed in ordinary crystals. For a comprehensive review of this fascinating subject we refer the reader to the recent book by Steurer and Douady [9].

The paper is structured as follows: in Section 1 we recall the generalized Delzant procedure; in Section 2 we introduce the quasilattice Q and we discuss its connection with the tiling; in Section 3 we construct the symplectic quasifolds M_b and M_r ; finally, in Section 4 we show that M_b and M_r are diffeomorphic but not symplectomorphic.

All pictures were drawn using the ZomeCAD software.

1 The Generalized Delzant Construction

We now recall from [6] the generalized Delzant construction. For the notion of quasifold and of related geometrical objects we refer the reader to the original article [6] and to [2], where some of the definitions were reformulated.

Let us recall what a *simple* convex polytope is.

Definition 1.1 (Simple polytope) A dimension n convex polytope $\Delta \subset (\mathbb{R}^n)^*$ is said to be *simple* if there are exactly n edges stemming from each vertex.

Let us next define the notion of quasilattice, introduced in [5]:

Definition 1.2 (Quasilattice) Let E be a real vector space. A *quasilattice* in E is the \mathbb{Z} -span of a set of \mathbb{R} -spanning vectors, Y_1, \dots, Y_d , of E .

Notice that $\text{Span}_{\mathbb{Z}}\{Y_1, \dots, Y_d\}$ is a lattice if and only if it admits a set of generators which is a basis of E .

Consider now a dimension n convex polytope $\Delta \subset (\mathbb{R}^n)^*$ having d facets. Then there exist elements X_1, \dots, X_d in \mathbb{R}^n and $\lambda_1, \dots, \lambda_d$ in \mathbb{R} such that

$$\Delta = \bigcap_{j=1}^d \{ \mu \in (\mathbb{R}^n)^* \mid \langle \mu, X_j \rangle \geq \lambda_j \}. \quad (1)$$

Definition 1.3 (Quasirational polytope) Let Q be a quasilattice in \mathbb{R}^n . A convex polytope $\Delta \subset (\mathbb{R}^n)^*$ is said to be *quasirational* with respect to Q if the vectors X_1, \dots, X_d in (1) can be chosen in Q .

We remark that each polytope in $(\mathbb{R}^n)^*$ is quasirational with respect to some quasilattice Q : just take the quasilattice that is generated by the elements X_1, \dots, X_d in (1). Notice that if X_1, \dots, X_d can be chosen in such a way that they belong to a lattice, then the polytope is rational in the usual sense. Before we go on to describing the generalized Delzant construction we recall what a *quasitorus* is.

Definition 1.4 (Quasitorus) Let $Q \subset \mathbb{R}^n$ be a quasilattice. We call *quasitorus* of dimension n the group and quasifold $D = \mathbb{R}^n/Q$.

For the definition of Hamiltonian action of a quasitorus on a symplectic quasifold we refer the reader to [6].

For the purposes of this article we will restrict our attention to the special case $n = 3$.

Theorem 1.5 (Generalized Delzant construction [6]) Let Q be a quasilattice in \mathbb{R}^3 and let $\Delta \subset (\mathbb{R}^3)^*$ be a simple convex polytope that is quasirational with respect to Q . Then there exists a 6 dimensional compact connected symplectic quasifold M and an effective Hamiltonian action of the quasitorus $D = \mathbb{R}^3/Q$ on M such that the image of the corresponding moment mapping is Δ .

Proof. Let us consider the space \mathbb{C}^d endowed with the standard symplectic form $\omega_0 = \frac{1}{2\pi i} \sum_{j=1}^d dz_j \wedge d\bar{z}_j$ and the action of the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$ given by

$$\begin{aligned} \tau: T^d \times \mathbb{C}^d &\longrightarrow \mathbb{C}^d \\ ((e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_d}), \underline{z}) &\longmapsto (e^{2\pi i \theta_1} z_1, \dots, e^{2\pi i \theta_d} z_d). \end{aligned}$$

This is an effective Hamiltonian action with moment mapping given by

$$\begin{aligned} J: \mathbb{C}^d &\longrightarrow (\mathbb{R}^d)^* \\ \underline{z} &\longmapsto \sum_{j=1}^d |z_j|^2 e_j^* + \lambda, \quad \lambda \in (\mathbb{R}^d)^* \text{ constant.} \end{aligned}$$

The mapping J is proper and its image is given by the cone $\mathcal{C}_\lambda = \lambda + Q$, where Q denotes the positive orthant of $(\mathbb{R}^d)^*$. Take now vectors $X_1, \dots, X_d \in Q$ and real numbers $\lambda_1, \dots, \lambda_d$ as in (1). Consider the surjective linear mapping

$$\begin{aligned} \pi: \mathbb{R}^d &\longrightarrow \mathbb{R}^2, \\ e_j &\longmapsto X_j. \end{aligned}$$

Consider the dimension 3 quasitorus $D = \mathbb{R}^3/Q$. Then the linear mapping π induces a quasitorus epimorphism $\Pi: T^d \longrightarrow D$. Define now N to be the kernel of the mapping Π and choose $\lambda = \sum_{j=1}^d \lambda_j e_j^*$. Denote by i the Lie algebra inclusion $\text{Lie}(N) \rightarrow \mathbb{R}^d$ and notice that $\Psi = i^* \circ J$ is a moment mapping for the induced action of N on \mathbb{C}^d . Then the quasitorus T^d/N acts in a Hamiltonian fashion on the compact symplectic quasifold $M = \Psi^{-1}(0)/N$. If we identify the quasitori D and T^d/N via the epimorphism Π , we get a Hamiltonian action of the quasitorus D whose moment mapping has image equal to $(\pi^*)^{-1}(\mathcal{C}_\lambda \cap \ker i^*) = (\pi^*)^{-1}(\mathcal{C}_\lambda \cap \text{im } \pi^*) = (\pi^*)^{-1}(\pi^*(\Delta))$ which is exactly Δ . This action is effective since the level set $\Psi^{-1}(0)$ contains points of the form $\underline{z} \in \mathbb{C}^d$, $z_j \neq 0$, $j = 1, \dots, d$, where the T^d -action is free. Notice finally that $\dim M = 2d - 2 \dim N = 2d - 2(d - 3) = 6$. \square

Remark 1.6 If we want to apply this construction to any simple convex polytope in $(\mathbb{R}^3)^*$, then there are two arbitrary choices involved. The first is the choice of a quasilattice Q with respect to which the polytope is quasirational, and the second is the choice of vectors X_1, \dots, X_d in Q that are orthogonal to the facets of Δ and inward-pointing as in (1).

2 Tilings and quasilattices

The purpose of this section is to introduce two quasilattices, R and Q , in \mathbb{R}^3 and to study how they relate to an Ammann tiling \mathcal{T} with fixed edge length $\sigma = \sqrt{1 + \frac{1}{\phi^2}}$, $\phi = \frac{1+\sqrt{5}}{2}$ being the golden section. We will be often using the following fundamental identity

$$\phi = 1 + \frac{1}{\phi}. \quad (2)$$

Ammann tilings are nonperiodic tilings of three dimensional space by so-called *golden rhombohedra*; rhombohedra are called golden when their facets are given by *golden rhombuses*, namely rhombuses with diagonals that are in the ratio of ϕ . There are two types of such rhombohedra which are called *oblate* and *prolate* (see Figures 1 and 2). For a review of Ammann tilings we refer the reader to [7, 9].

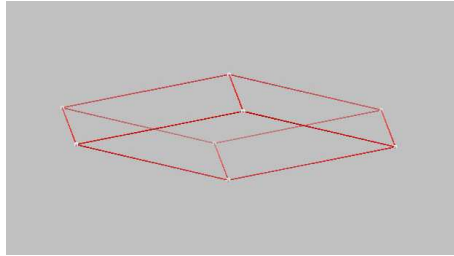


Figure 1: The oblate rhombohedron

Let R be the quasilattice in $(\mathbb{R}^3)^*$ that is generated by the vectors

$$\begin{aligned} V_1 &= (\phi - 1, 1, 0) \\ V_2 &= (0, \phi - 1, 1) \\ V_3 &= (1, 0, \phi - 1) \\ V_4 &= (1 - \phi, 1, 0) \\ V_5 &= (0, 1 - \phi, 1) \\ V_6 &= (1, 0, 1 - \phi). \end{aligned}$$

These six vectors and their opposites point to the twelve vertices of an icosahedron that is inscribed in the sphere of radius σ (see Figures 3 and 4); they are the only vectors of the quasilattice that have norm σ . One can check that these vectors form a minimal set of generators of R and that R is dense in $(\mathbb{R}^3)^*$.

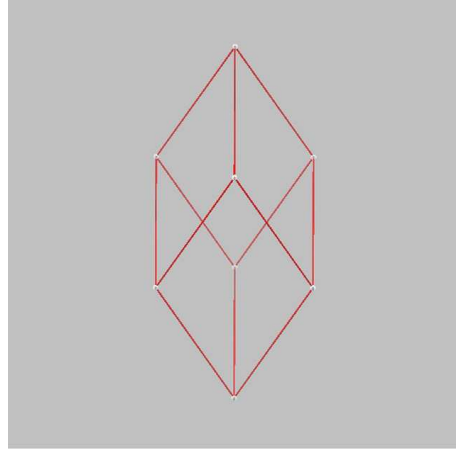
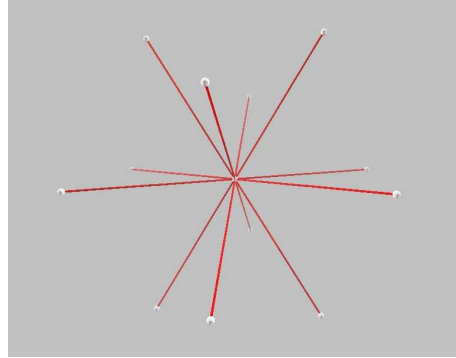


Figure 2: The prolate rhombohedron

Figure 3: The vectors $\pm V_1, \dots, \pm V_6$

Consider now the two following golden rhombohedra: the oblate rhombohedron Δ_b^o , having nonparallel edges V_4, V_5, V_6 , and the prolate rhombohedron Δ_r^o , having nonparallel edges V_1, V_2, V_3 .

The following proposition fully describes how the tiling and the quasilattice R relate. Denote by AB one edge of the tiling \mathcal{T} . From now on we will choose our coordinates so that $A = O$ and so that $B - A$ is parallel to V_1 with the same orientation.

Proposition 2.1 Let \mathcal{T} be an Ammann tiling with edges of length σ . Each vertex of the tiling lies in the quasilattice R . Moreover, for each oblate rhombohedron Δ_b in \mathcal{T} (respectively prolate rhombohedron Δ_r in \mathcal{T}) there is a rigid motion ρ , given by the composition of a translation with a transformation of the icosahedral group, such that $\rho(\Delta_b)$ is Δ_b^o (respectively $\rho(\Delta_r)$ is Δ_r^o).

Proof. Consider first the icosahedron with its twenty pairwise parallel facets. To each pair of parallel facets there correspond two oblate rhombohedra, one the translate of

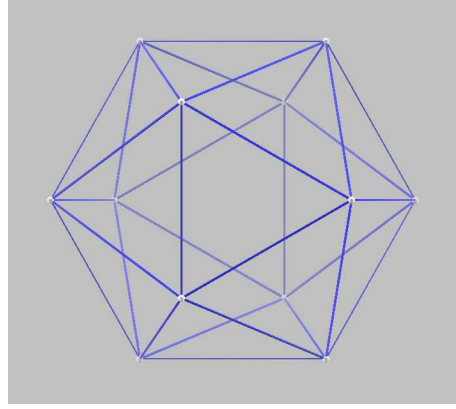


Figure 4: The icosahedron

the other, and two prolate rhombohedra, also one the translate of the other. Pick one representative for each such couple. This gives a total of ten oblate rhombohedra and ten prolate rhombohedra. Each of the ten oblate rhombohedra can be mapped to Δ_b^o via a transformation of the icosahedral group, and in the same way each of the ten prolate rhombohedra can be mapped to Δ_r^o .

Now, let C be a vertex of the tiling that is different from 0 and the above vertex B . We can join B to C with a broken line made of subsequent edges of the tiling. We denote the vertices of the broken line thus obtained by $T_0 = A, T_1 = B, \dots, T_j, \dots, T_m = C$. Since the tiles are oblate and prolate rhombohedra, each vector $Y_j = T_j - T_{j-1}$ is one of the vectors $\pm V_k, k = 1, \dots, 6$. Therefore we have that $C - A = T_m - T_0 = Y_m + \dots + Y_1$. This implies that the vertex C lies in R , that each oblate rhombohedron having C as vertex is the translate of one of the ten oblate rhombohedra described above and that each prolate rhombohedron having C as vertex is the translate of one of the ten prolate rhombohedra described above. We can therefore conclude that, for each oblate rhombohedron Δ_b having C as vertex, there exists a rigid motion ρ , given by the composition of a translation with a transformation of the icosahedral group, such that $\rho(\Delta_b) = \Delta_b^o$. The same is true for the prolate rhombohedra. \square

Recall that the rhombic triacontahedron is a convex polyhedron with thirty facets, each given by a golden rhombus. It is first found in Kepler's writings [4]. The long diagonals of its rhombic facets are the edges of an icosahedron while the short diagonals are the edges of a dodecahedron. Consider now the triacontahedron S that has $\pm V_i, i = 1, \dots, 6$, among its vertices. Proposition 2.1 implies that all of the facets of the Ammann tiling are parallel to the thirty, pairwise parallel, facets of the triacontahedron S . Therefore, a quasilattice with respect to which all of the rhombohedra of the tiling are quasirational must contain vectors that are normal to the thirty facets of the triacontahedron. We choose as generators of our quasilattice Q the thirty vertices of the icosidodecahedron dual to S (see figure 6 and figure 7); this is an icosidodecahedron inscribed in the sphere of radius 1. Among these thirty unit vectors we select the

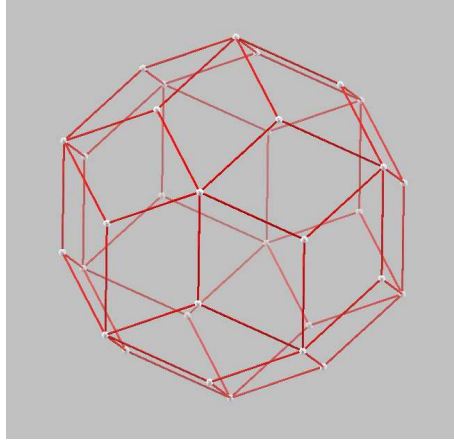


Figure 5: The triacontahedron

following six vectors, which give a minimal set of generators of Q :

$$\begin{aligned} U_1 &= \frac{1}{2}(1, \phi - 1, \phi) \\ U_2 &= \frac{1}{2}(\phi, 1, \phi - 1) \\ U_3 &= \frac{1}{2}(\phi - 1, \phi, 1) \\ U_4 &= \frac{1}{2}(-1, \phi - 1, \phi) \\ U_5 &= \frac{1}{2}(\phi, -1, \phi - 1) \\ U_6 &= \frac{1}{2}(\phi - 1, \phi, -1). \end{aligned}$$

It can be seen that Q is dense in \mathbb{R}^3 and that the thirty unit vectors pointing to the vertices of the icosidodecahedron are the only unit vectors in the quasilattice Q .

The quasilattice Q is invariant under icosahedral symmetries (as well as the quasilattice R) and, by construction, all of the rhombohedra of the tiling are quasirational with respect to Q . We will see in Theorem 3.1 that icosahedral symmetry is essential for the uniqueness, from the differentiable viewpoint, of the quasifold corresponding to the tiling.

3 The Tiling from a Symplectic Viewpoint

In this section we perform the Delzant construction to obtain symplectic quasifolds that can be associated to the oblate and prolate rhombohedra of an Ammann tiling having edge length σ .

Let us consider the quasilattice Q that we introduced in Section 2. As we have seen, all of the rhombohedra of our tiling are quasirational with respect to Q .

We begin by considering the oblate rhombohedron Δ_b^o which has one of its vertices at the origin and is determined by the three non-parallel vectors V_4, V_5, V_6 . This simple polytope has 6 facets. For our construction we choose the 6 vectors given by $X_1 = U_1$, $X_2 = U_2$, $X_3 = U_3$, $X_4 = -U_1$, $X_5 = -U_2$ and $X_6 = -U_3$. Then the corresponding

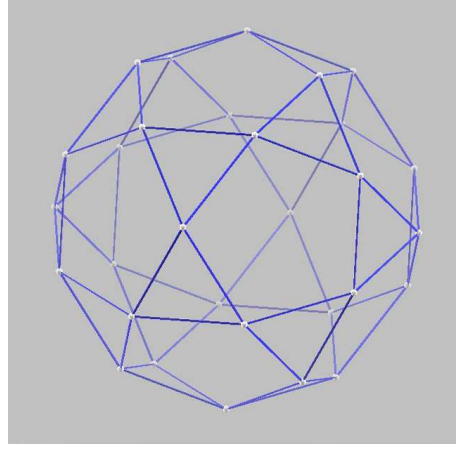
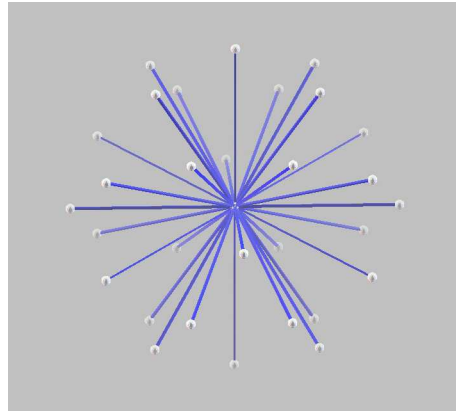


Figure 6: The icosidodecahedron

Figure 7: The star of unit vectors of the quasilattice Q

coefficients are given by $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = \lambda_5 = \lambda_6 = -\frac{1}{\phi}$. Take now the surjective linear mapping defined by

$$\begin{aligned} \pi: \mathbb{R}^6 &\rightarrow \mathbb{R}^3 \\ e_i &\mapsto X_i \end{aligned}$$

Its kernel, \mathfrak{n} , is the 3-dimensional subspace of \mathbb{R}^6 that is spanned by $e_1 + e_4, e_2 + e_5$ and $e_3 + e_6$. It is the Lie algebra of $N = \{\exp(X) \in T^6 \mid X \in \mathbb{R}^6, \pi(X) \in Q\}$. If Ψ_b is the moment mapping of the induced N -action, then

$$\begin{aligned} \Psi_b: \mathbb{C}^6 &\rightarrow (\mathbb{R}^3)^* \\ z &\mapsto \left(|z_1|^2 + |z_4|^2 - \frac{1}{\phi}, |z_2|^2 + |z_5|^2 - \frac{1}{\phi}, |z_3|^2 + |z_6|^2 - \frac{1}{\phi} \right) \end{aligned}$$

Therefore $\Psi_b^{-1}(0) = S_b^3 \times S_b^3 \times S_b^3$, where S_b^3 is the sphere in \mathbb{R}^4 centered at the origin with radius $b = \frac{1}{\sqrt{\phi}}$. In order to compute the group N we need the following linear

relations between the generators of the quasilattice Q :

$$\begin{pmatrix} U_4 \\ U_5 \\ U_6 \end{pmatrix} = \begin{pmatrix} 1 & -\phi & 1 \\ 1 & 1 & -\phi \\ -\phi & 1 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad (3)$$

Then a straightforward computation gives that

$$N = \left\{ \exp(X) \in T^6 \mid X = (r + \phi h, s + \phi k, t + \phi l, r, s, t), r, s, t \in \mathbb{R}, h, k, l \in \mathbb{Z} \right\}.$$

We can think of

$$S^1 \times S^1 \times S^1 = \{ \exp(X) \in T^6 \mid X = (r, s, t, r, s, t), r, s, t \in \mathbb{R} \} \quad (4)$$

as being naturally embedded in N . The quotient group

$$\Gamma = \frac{N}{S^1 \times S^1 \times S^1}$$

is discrete. In conclusion, the symplectic quotient M_b is given by

$$M_b = \frac{\Psi_b^{-1}(0)}{N} = \frac{S_b^3 \times S_b^3 \times S_b^3}{N} = \frac{S_b^2 \times S_b^2 \times S_b^2}{\Gamma},$$

where S_b^2 is the sphere in \mathbb{R}^3 centered at the origin with radius b . The quasitorus $D^3 = \mathbb{R}^3/Q$ acts on M_b in a Hamiltonian fashion, with image of the corresponding moment mapping given exactly by the oblate rhombohedron Δ_b^o .

Consider now the prolate rhombohedron Δ_r^o that has one vertex in the origin and is determined by the three nonparallel vectors V_1, V_2, V_3 . We now choose the vectors given by $X_1 = U_4, X_2 = U_5, X_3 = U_6, X_4 = -U_4, X_5 = -U_5$ and $X_6 = -U_6$. Then the corresponding coefficients are given by $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = \lambda_5 = \lambda_6 = -1$. It is immediate to check that we obtain the same Lie algebra \mathfrak{n} as in the case of the oblate rhombohedron. In order to see what happens to the corresponding group we need here the inverse relations:

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{\phi} \\ \frac{1}{\phi} & 1 & 1 \\ 1 & \frac{1}{\phi} & 1 \end{pmatrix} \begin{pmatrix} U_4 \\ U_5 \\ U_6 \end{pmatrix} \quad (5)$$

To write the relations in this form we used the fundamental identity 2. This identity also implies that we obtain the same group N as in the case of the oblate rhombohedron.

The moment mapping Ψ_r is given by:

$$\begin{aligned} \Psi_r: \quad \mathbb{C}^6 &\longrightarrow (\mathbb{R}^3)^* \\ \underline{z} &\longmapsto (|z_1|^2 + |z_4|^2 - 1, |z_2|^2 + |z_5|^2 - 1, |z_3|^2 + |z_6|^2 - 1). \end{aligned}$$

Therefore

$$M_r = \frac{\Psi_r^{-1}(0)}{N} = \frac{S^3 \times S^3 \times S^3}{N} = \frac{S^2 \times S^2 \times S^2}{\Gamma},$$

where S^3 and S^2 are the unit spheres centered at the origin, of dimension 3 and 2 respectively. The quasifold M_r is acted on by the same quasitorus $D^3 = \mathbb{R}^3/Q$ that we

obtained for the oblate rhombohedron. This action is Hamiltonian and the image of the corresponding moment mapping is exactly the prolate rhombohedron Δ_r^o .

Let us remark that M_b and M_r are both global quotients and that this defines their quasifold structures.

Remark now that, by Proposition 2.1, *each* of the oblate and prolate rhombohedra in the tiling can be obtained from Δ_b^o and Δ_r^o respectively by a transformation of the icosahedral group composed with a translation. We can then prove the following

Theorem 3.1 Consider an Ammann tiling having edge length σ . Then the compact connected symplectic quasifold corresponding to each oblate rhombohedron in the tiling is given by M_b , while the compact connected symplectic quasifold corresponding to each prolate rhombohedron is given by M_r .

Proof. Observe that, for each oblate rhombohedron, there exists a transformation P in the icosahedral group that leaves the quasilattice Q invariant, that sends the orthogonal vectors relative to the chosen oblate rhombohedron to the orthogonal vectors relative to Δ_b^o , and such that the dual transformation P^* sends Δ_b^o to a translate of the chosen oblate rhombohedron. The same reasoning applies to the prolate rhombohedra of the tiling. This implies that the reduced space corresponding to each oblate rhombohedron of the tiling, with the choice of orthogonal vectors and quasilattice specified above, is exactly M_b . This yields a unique symplectic quasifold, M_b , for all the oblate rhombohedra in the tiling. In the same way we prove that we obtain a unique symplectic quasifold, M_r , for all the prolate rhombohedra in the tiling. \square

4 Symplectotype and Diffeotype of the Tiles

The purpose of this section is to prove the following

Theorem 4.1 The quasifolds M_b and M_r are diffeomorphic but not symplectomorphic.

Before we proceed with the proof let us describe a special atlas for the quasifold M_b . The charts of this atlas are indexed by the vertices of the polytope: in our case we find an atlas given by eight charts, each of which corresponds to a vertex of the oblate rhombohedron. Consider for example the origin: it is given by the intersection of the facets whose orthogonal vectors are X_1, X_2 and X_3 . We will label the corresponding chart by the index 1. Let B_b be the ball in \mathbb{C} of radius b , namely

$$B_b = \{z \in \mathbb{C} \mid |z| < b\}.$$

Consider the following mapping, which gives a slice of $\Psi^{-1}(0)$ transversal to the N -orbits

$$\begin{array}{ccc} B_b \times B_b \times B_b & \xrightarrow{t_1} & \{\underline{z} \in \Psi^{-1}(0) \mid z_4 \neq 0, z_5 \neq 0, z_6 \neq 0\} \\ (z_1, z_2, z_3) & \longmapsto & (z_1, z_2, z_3, \sqrt{b^2 - |z_1|^2}, \sqrt{b^2 - |z_2|^2}, \sqrt{b^2 - |z_3|^2}) \end{array}.$$

This induces the homeomorphism

$$\begin{array}{ccc} (B_b \times B_b \times B_b)/\Gamma_1 & \xrightarrow{\tau_1} & U_1 \\ [\underline{z}] & \longmapsto & [t_1(\underline{z})] \end{array},$$

where the open subset U_1 of M_b is the quotient

$$\{\underline{z} \in \Psi^{-1}(0) \mid z_4 \neq 0, z_5 \neq 0, z_6 \neq 0\}/N$$

and the discrete group Γ_1 is given by $\Gamma_1 \simeq N \cap (S^1 \times S^1 \times S^1 \times \{1\} \times \{1\} \times \{1\})$, hence

$$\Gamma_1 = \exp \{(\phi h, \phi k, \phi l) \mid h, k, l \in \mathbb{Z}\}. \quad (6)$$

The triple $(U_1, \tau_1, (B_b \times B_b \times B_b)/\Gamma_1)$ is a chart of M_b . Analogously, we can construct seven other charts, corresponding to the remaining vertices of the oblate rhombohedron, each of which is characterized by a different combination of the variables. One can show that these eight charts are compatible and give an atlas of M_b .

Now let us denote by p_b and p_r the projections

$$p_b : S_b^2 \times S_b^2 \times S_b^2 \rightarrow M_b \quad \text{and} \quad p_r : S_r^2 \times S_r^2 \times S_r^2 \rightarrow M_r.$$

Denote by V_n the open subset of S_b^2 given by S_b^2 minus the south pole and by V_s the open subset of S_b^2 given by S_b^2 minus the north pole. Then, on $\Psi^{-1}(0)$, consider the action of $S^1 \times S^1 \times S^1$ given by (4). We obtain

$$V_n \times V_n \times V_n = \{\underline{z} \in \Psi^{-1}(0) \mid z_4 \neq 0, z_5 \neq 0, z_6 \neq 0\}/(S^1 \times S^1 \times S^1)$$

and

$$U_1 = (V_n \times V_n \times V_n)/\Gamma.$$

We have the following commutative diagram:

$$\begin{array}{ccc} B_b \times B_b \times B_b & \xrightarrow{t_1} & \{\underline{z} \in \Psi^{-1}(0) \mid z_4 \neq 0, z_5 \neq 0, z_6 \neq 0\} \\ \downarrow & & \downarrow \\ B_b \times B_b \times B_b & \xrightarrow{\tilde{\tau}_1} & V_n \times V_n \times V_n \\ p_1 \downarrow & & \downarrow p_b \\ (B_b \times B_b \times B_b)/\Gamma_1 & \xrightarrow{\tau_1} & U_1 \end{array} \quad (7)$$

The mapping $\tilde{\tau}_1$ is induced by the diagram and can be written as $\tau_n \times \tau_n \times \tau_n$, with $\tau_n : B_b \rightarrow V_n$. Observe that the mapping

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & V_n \\ w & \longmapsto & [\tau_n(b w / \sqrt{1 + |w|^2})] \end{array}$$

is just the stereographic projection from the north pole. We denote by τ_s the analogous mapping $\tau_s : B_b \rightarrow V_s$. The two charts (B_b, τ_n, V_n) and (B_b, τ_s, V_s) give a symplectic atlas of S_b^2 , whose standard symplectic structure is induced by the standard symplectic structure on B_b . Analogously, at a local level, the symplectic structure of the quotient M_b is induced by the standard symplectic structure on $B_b \times B_b \times B_b$.

We have already seen that the quasifold M_b is a global quotient of a product of three 2-spheres by the discrete group Γ . We remark that the atlas above is the quotient by

Γ of the atlas of the product of three spheres, given by the eight triples $V_n \times V_n \times V_n$, $V_n \times V_n \times V_s$, $V_n \times V_s \times V_n$, $V_s \times V_n \times V_n$, $V_n \times V_s \times V_s$, $V_s \times V_n \times V_s$, $V_s \times V_s \times V_n$, $V_s \times V_s \times V_s$.

We are now ready to prove our result:

Proof of Theorem 4.1. Let us begin by showing that M_b and M_r are diffeomorphic. The natural Γ -equivariant diffeomorphism $f^\dagger: S_b^2 \times S_b^2 \times S_b^2 \rightarrow S_r^2 \times S_r^2 \times S_r^2$ induces a homeomorphism $f: M_b \rightarrow M_r$; in general, a homeomorphism between two global quotients that is induced by an equivariant diffeomorphism of the manifolds turns out to be a quasifold diffeomorphism [2, Definition A.2].

Let us now show that M_b and M_r are not symplectomorphic. Denote by ω_b and ω_r the symplectic forms of M_b and M_r respectively. Suppose that there is a symplectomorphism $h: M_b \rightarrow M_r$, namely a diffeomorphism h such that $h^*(\omega_r) = \omega_b$. We prove that this implies that the homeomorphism $h: M_b \rightarrow M_r$ lifts to a symplectomorphism $\tilde{h}: S_b^2 \times S_b^2 \times S_b^2 \rightarrow S_r^2 \times S_r^2 \times S_r^2$, leading thus to a contradiction: such symplectomorphism cannot exist, since the two manifolds have different symplectic volumes. To start with recall from [2, Remark 2.9] that, to each point $m \in M_b$, one can associate the groups Γ_m and $\Gamma_{h(m)}$. The definition of diffeomorphism implies that these two groups are isomorphic. Let $n_b \in S_b^2$ be the north pole and take $m_0 = p_b(n_b \times n_b \times n_b)$. Then, since $\Gamma_m \simeq \Gamma_{h(m)}$, without loss of generality the point $h(m_0)$ can be taken to be $p_r(n_r \times n_r \times n_r)$, where $n_r \in S_r^2$ is the north pole. Consider the chart U_1 that we constructed above. Then, by definition of quasifold diffeomorphism [2, Definition A.23] and [2, Remark A.24], there exists an open subset $U \subset U_1$ such that $m_0 \in U$ and $h \circ \tau_1^{-1}: \tau_1^{-1}(U) \rightarrow h(U)$ is a diffeomorphism of the universal covering models induced by $\tau_1^{-1}(U) \subset B_b \times B_b \times B_b / \Gamma_1$ and $h(U) \subset M_r$ respectively. Moreover, by [2, Proposition A.9], any open subset $W \subset U$ enjoys the same property. We can choose $W_0 \subset U_1$ such that $\tilde{W}_0 = (\tau_1 \circ p_1)^{-1}(W_0)$ is a product of three balls. In particular, \tilde{W}_0 is simply connected. Denote now by $\tilde{W}_{r,0} = (p_r)^{-1}(h(W_0))$; this is an open subset of $S_r^2 \times S_r^2 \times S_r^2$, which is also connected, due to the action of Γ on $S_r^2 \times S_r^2 \times S_r^2$. Denote by $\tilde{W}_{r,0}^\#$ its universal covering. Now consider a point $\underline{z}^1 \in B_b \times B_b \times B_b$ such that $z_1^1 \neq 0$, $z_2^1 \neq 0$ and $z_3^1 \neq 0$ and let $m = (\tau_1 \circ p_1)(\underline{z}^1)$. For the sequel it is crucial to remark that, because of the action of Γ_1 given in (6), any Γ_1 -invariant open subset of $B_b \times B_b \times B_b$ that contains the point \underline{z}^1 , contains also the product of circles $\{(z_1, z_2, z_3) \in B_b \times B_b \times B_b \mid |z_1| = |z_1^1|, |z_2| = |z_2^1|, |z_3| = |z_3^1|\}$. Hence, for each point $(\tau_1 \circ p_1)(t\underline{z}^1)$ with $t \in [0, 1]$, we can find an open subset $W_t \subset U_1$, containing that point, such that the homeomorphism $\tau_1^{-1} \circ h$, restricted to $\tau_1^{-1}(W_t)$, is a diffeomorphism, and $(\tau_1 \circ p_1)^{-1}(W_t)$ is the product of three open annuli. We can cover the curve by a finite number of these W_t 's: W_0, W_1, \dots, W_s , with $W_j \cap W_{j+1} \neq \emptyset$. Notice that $(\tau_1 \circ p_1)^{-1}(W_j \cap W_{j+1})$, $j = 0, \dots, s-1$, is itself a product of three open annuli. The subsets $\tilde{W}_j = (\tau_1 \circ p_1)^{-1}(W_j)$ and $\tilde{W}_{r,j} = (p_r)^{-1}(h(W_j))$ are open and connected.

We divide the remaining part of the proof in subsequent steps:

Step 1: consider first W_0 . Since the isotropy of Γ_1 at 0 is the whole Γ_1 , we can apply [2, Lemma 6.2]. We find that $\tilde{W}_{r,0}$ is itself simply connected and that the homeomorphism $h \circ \tau_1$ lifts to a diffeomorphism $\tilde{h}_0: \tilde{W}_0 \rightarrow \tilde{W}_{r,0}$.

Step 2: consider the homeomorphism $h_1 = h \circ \tau_1$ defined on $\tau_1^{-1}(W_1)$. By construction

h_1 is a diffeomorphism of the universal covering models of the induced models. We find the following diagram:

$$\begin{array}{ccc}
 W_1^\# & \xrightarrow{h_1^\#} & W_{r,1}^\# \\
 \pi_1 \downarrow & & \downarrow \rho_1 \\
 \tilde{W}_1 & & \tilde{W}_{r,1} \\
 p_1 \downarrow & & \downarrow q_1 \\
 \tau_1^{-1}(W_1) & \xrightarrow{h_1} & h(W_1)
 \end{array}$$

Consider the restriction of h_1 to $\tau_1^{-1}(W_0 \cap W_1)$. This restriction admits a lift, given by the restriction of $h_1^\#$ to $(\pi_1 \circ p_1)^{-1}(\tau_1^{-1}(W_0 \cap W_1))$. Furthermore, by Step 1, the restriction of h_1 admits another lift, defined on $p_1^{-1}(\tau_1^{-1}(W_0 \cap W_1))$, which is the restriction of \tilde{h}_0 . Therefore, by [2, Lemma 6.3], the restriction of $\rho_1 \circ h_1^\#$ to $(\pi_1 \circ p_1)^{-1}(\tau_1^{-1}(W_0 \cap W_1))$ descends to a diffeomorphism defined on $p_1^{-1}(\tau_1^{-1}(W_0 \cap W_1))$.

Step 3: we consider $W_0 \cap W_1 \subset W_1$ and we apply [2, Lemma 6.5] to the homeomorphism $h \circ \tau_1$ defined on $\tau_1^{-1}(W_1)$. We deduce that $h \circ \tau_1$ is a diffeomorphism of the model $(\tau_1 \circ p_1)^{-1}(W_1)/\Gamma_1$ with the model induced by $h(W_1) \subset M_r$.

Step 4: we apply Step 3 to the other successive intersections. We find that $h \circ \tau_1$ is a diffeomorphism of the model $(\tau_1 \circ p_1)^{-1}(\cup_{i=1}^k W_i)/\Gamma_1$ with the model induced by $h(\cup_{i=1}^k W_i) \subset M_r$. Remark now that a slight modification of the above argument applies to any choice of point $\underline{z}^1 \in B_b \times B_b \times B_b$, $\underline{z}_1 \neq 0$.

Let $\epsilon > 0$ be arbitrarily small. Consider the product of closed balls $\overline{B}_{b-\epsilon} \times \overline{B}_{b-\epsilon} \times \overline{B}_{b-\epsilon}$. This, by Step 4, can be covered by a finite number of connected open subsets of the kind $(\tau_1 \circ p_1)^{-1}(\cup_{i=1}^k W_i)/\Gamma_1$, whose intersection is a product of three balls centered at the origin. Now [2, Lemma A.3], which guarantees the uniqueness of the lift up to the action of Γ , implies that the homeomorphism h admits a lift to $\tilde{\tau}_1(\overline{B}_{b-\epsilon} \times \overline{B}_{b-\epsilon} \times \overline{B}_{b-\epsilon})$. This in turn implies that $h: U_1 \rightarrow h(U_1)$ admits a lift

$$\tilde{h}_1: V_n \times V_n \times V_n \rightarrow p_r^{-1}(h(U_1)).$$

We apply the same argument to the other eight charts. These charts intersect on the dense connected open subset where the action of the quasitorus D_b is free. By the uniqueness of the lift, ([2, Lemma A.3]), we obtain a global lift $\tilde{h}: S_b^2 \times S_b^2 \times S_b^2 \rightarrow S_r^2 \times S_r^2 \times S_r^2$. Moreover, since diagram (7) preserves the symplectic structures, we have that \tilde{h} is a symplectomorphism between $S_b^2 \times S_b^2 \times S_b^2$ to $S_r^2 \times S_r^2 \times S_r^2$, which is impossible. \square

In conclusion, there is a *unique* quasifold structure that is naturally associated to any Ammann tiling with fixed edge length, and two distinct symplectic structures that distinguish the oblate and the prolate rhombohedra.

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DIPARTIMENTO DI MATEMATICA APPLICATA "G. SANSONE", UNIVERSITÀ DI FIRENZE, VIA S. MARTA 3, 50139 FIRENZE, ITALY, fiammetta.battaglia@unifi.it

AND

DIPARTIMENTO DI MATEMATICA "U. DINI", PIAZZA Ghiberti 27, 50122 FIRENZE, ITALY, elisa.prato@unifi.it